Topology and its Applications 48 (1992) 19-23
North-Holland

Sufficient conditions for the incompressibility of the boundary of an n-relator 3-manifold

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Received 1 April 1991
Revised 29 October 1991

Abstract

Lei Fengchun, Sufficient conditions for the incompressibility of the boundary of an n-relator 3-manifold, Topology and its Applications 48 (1992) 19-23.

In this paper we give sufficient conditions for the incompressibility of the boundary of an n-relator 3-manifold. The conditions are those conjectured to be sufficient by Przytycki, with one additional indispensable condition.

Keywords: Incompressibility, bind, coplanar with.

AMS (MOS) Subj. Class.: 57M99.

1. Introduction

In 1983, Przytycki [5] first gave sufficient conditions for the boundary of a 1-relator 3-manifold to be incompressible, and in 1984, Jaco [2] extended Przytycki's result by using a geometric approach. But there exist examples (see [5]) which indicate that a direct generalization of the above results to the case of n-relator 3-manifolds is not possible. In 1987, Przytycki [6] proposed a set of conditions which might be sufficient.

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* Supported by the National Natural Science Foundation of China.

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Przytycki's conjecture. Let $C=\{J_1,\dots,J_n\}$ be a family of 2-sided, pairwise disjoint, simple closed curves in the boundary of a handlebody H (with genus $k>0$). Assume that the following conditions are satisfied (see Section 2 for necessary definitions):

(0) $\partial H - C$ is incompressible in H,

(1) for each j, $\partial H-(C-J_j)$ is compressible in H (or equivalently, the family of elements of $\pi_1(H)=F_k$ represented by $C-J_j$ does not bind the free group F_k),

(2) for each pair j, s $(j\neq s)$, $C-\{J_j,J_s\}$ does not bind any free factor F_{k-1} of $F_k=F_{k-1}\times F_1$,

$\vdots$

(p) no $(n-p)$-element subfamily of C binds a free factor F_{k-p+1} of $F_k=F_{k-p+1}\times F_{p-1}$,

$\vdots$

$(n-1)$ no curve J_j of C binds a free factor F_{k-n+2} of $F_k=F_{k-n+2}\times F_{n-2}$.

Then the n-relator 3-manifold H_C has incompressible boundary, or it is equal to D^3.

In [6] Przytycki proved his conjecture for $n=1,2,$ and 3. When $n>3$, he had examples to show that all the assumptions in his conjecture are necessary.

In this paper we shall prove Przytycki's conjecture assuming the following additional condition:

(n) for each j, J_j is not contained in the normal subgroup of $\pi_1(H)=F_k$ generated by any $(n-2)$-subfamily of $C-J_j$.

We shall also show that some such additional condition is required.

2. Preliminaries

We work in the PL-category and use scc as an abbreviation of simple closed curve (or curves).

Definition 2.1. Let M be a 3-manifold and S a surface which is either properly embedded in M or contained in ∂M. We say that S is *compressible* (in M) if one of the following conditions is satisfied:

(1) S is a 2-sphere which bounds a 3-cell in M, or

(2) S is a 2-cell and either $S\subset\partial M$ or there is a 3-cell $X\subset M$ with $\partial X\subset S\cup\partial M$, or

(3) there is a 2-cell $D\subset M$ with $D\cap S=\partial D$ and with ∂D not contractible in S.

In case (3), D is also called a *compressing disk* for S (in M). We say that S is *incompressible* if it is not compressible.

Definition 2.2. Let M be a 3-manifold and J a 2-sided scc on ∂M. Let A_J be a regular neighbourhood of J in ∂M, (D^3,A) a 3-cell with an annulus $A\subset\partial D^3$, and h a homeomorphism $A_J\to A$. Then the 3-manifold $(M,A_J)\cup_h(D^3,A)$ is denoted M_J. If $C=\{J_1,\dots,J_n\}$ is a collection of pairwise disjoint, 2-sided sccs on ∂M, then we denote $M_C = (\ldots((M_{J_1})_{J_2})\ldots)_{J_n}$. In particular, when M is a handlebody H with genus k (>0), H_C is called an *n-relator* 3*-manifold.*

Clearly, the definition of M_C does not depend on the order of the J_i.

Definition 2.3. Let $C = \{J_1, \ldots, J_n\}$ be a family of pairwise disjoint 2-sided sccs on a surface S. We say that an scc $J \subset S - C$ is coplanar with C if J cuts a disk with holes from S cut open along C (i.e., $S - C$).

Definition 2.4. Let $W \subset F_k$ be a set of cyclic words in the free group F_k with a basis X. The *incidence graph* $J(W)$ is the graph whose vertices are in 1-1 correspondence with the nontrivial words in W, with an edge joining vertices w_1 and w_2 if there exists $x \in X$ such that x or x^{-1} lies in w_1 and x or x^{-1} lies in w_2. W is *connected with respect to the basis* X if $J(W)$ is connected, and is *connected* if it is connected with respect to each basis of F_k. If the set W of cyclic elements is not contained in any proper free factor of F_k and if W is connected, we say that W *binds* F_k.

For convenience, we shall refer to disks with holes as "planar surfaces". We shall also abuse notation slightly by using the symbol C, which represents a family of closed curves in the 3-manifold M, also to represent the corresponding elements of $\pi_1(M)$ when this causes no confusion.

The following two lemmas will be used in our proof:

Lemma 2.5 (Due to Przytycki [6]). *Let* $C = \{J_1, \ldots, J_n\}$ *be a family of pairwise disjoint,* 2*-sided sccs on* ∂M *and let the following conditions be satisfied*:

(i) $\partial M - C$ *is incompressible in* M,

(ii) *for each* j, $\partial M - (C - J_j)$ *is compressible in* M,

(iii) *for each* j, *a compressing disk from* (ii), *say* D, *can be chosen in such a way that* ∂D *is not coplanar with* $C - J_j$.

Then M_C *has incompressible boundary or it is equal to* D^3.

Lemma 2.6 (Due to Lyon [4]). *Let* C *be a family of pairwise disjoint sccs on the boundary of a handlebody* H. *Then* $S = \mathrm{cl}(\partial H - N(C))$ *is incompressible if and only if* C *binds* $\pi_1(H)$ *and no curve in* C *is contractible in* ∂H.

3. The proof

Theorem 3.1. *Let* $C = \{J_1, \ldots, J_n\}$ *be a family of pairwise disjoint,* 2*-sided sccs on the boundary of a handlebody* H *with genus* $k > 0$. *Suppose the following conditions are satisfied*:

(I) *the conditions* (0)-($n-1$) *in Przytycki's conjecture, and*

(II) *for each* j, J_j *is not contained in the normal subgroup of* $\pi_1(H) = F_k$ *generated by any* ($n-2$)*-subfamily of* $C - J_j$.

Then the n-relator 3*-manifold* H_C *has incompressible boundary, or it is equal to* D^3.

Proof. Without condition (II), the theorem was proved for $n = 1, 2$ and 3 by Przytycki (see [5, 6]). Here we only need to consider the case of $n > 3$.

From assumption (I)(0) we easily have

Assertion 1. *No curve in C is contractible in H.*

By (I)(1), for each j, $\partial H - (C - J_j)$ is compressible in H. If for each J_j, a compressing disk of $\partial H - (C - J_j)$, say D, can be chosen in such a way that ∂D is not coplanar with $C - J_j$, then the given conditions (I)(0) and (I)(1) and Lemma 2.5 imply that H_C has incompressible boundary or it is equal to D^3, the proof is already complete. Henceforth, it is sufficient to consider the other case.

In the following, without loss of generality, we suppose that

$$\text{each compressing disk of } \partial H - (C - J_n) \text{ in } H \text{ has boundary coplanar with } C - J_n. \tag{$*$}$$

Let S denote the surface ∂H cut open along $C - J_n$, and let J_i' and J_i'' denote the two boundary components of S corresponding to the curve J_i, $1 \leqslant i \leqslant n-1$. Let Δ be a compressing disk of $\partial H - (C - J_n)$. By ($*$), $\partial\Delta$ is coplanar with $C - J_n$, that is, $\partial\Delta$ and a subset (with at least two elements, by Assertion 1) of $\partial S = \{J_j' \colon 1 \leqslant i \leqslant n-1\} \cup \{J_j'' \colon 1 \leqslant i \leqslant n-1\}$ bound a planar surface S^*. First we consider the case that $\partial\Delta$ does not separate ∂H (therefore Δ does not separate H). In this situation, the curves of ∂S^* cannot all be paired, thus there exists some J_{i_0}' (or J_{i_0}'') $\in \partial S^*$ but J_{i_0}'' (or J_{i_0}') $\notin \partial S^*$. $S^* \cup \Delta$ is an embedded planar surface in H, hence J_{i_0} is contained in the normal subgroup of $\pi_1(H)$ generated by the subfamily $\{J_i \colon 1 \leqslant i \leqslant n-1,\ i \neq i_0\}$ of C. This contradicts assumption (II). So we have

Assertion 2. *Every compressing disk in H of $\partial H - (C - J_n)$ separates H.*

Thus we know that Δ does separate H, therefore the curves contained in ∂S^* are all paired, that is, if j_i' (or J_i'') $\in \partial S^*$, then J_i'' (or J_i') $\in \partial S^*$. So, without loss of generality we assume that

$$\partial S^* = \partial\Delta \cup \{J_{i_0+1}', J_{i_0+1}'', \ldots, J_{n-1}', J_{n-1}''\},$$

where $i_0 \geqslant 1$ (otherwise $\partial\Delta$ is contractible on S), and Δ divides H into two handlebodies H_1 and H_2, with genus k_1 and k_2, respectively, and $k_1 = k - n + i_0 + 1$, $k_2 = n - 1 - i_0$, $k_1, k_2 > 0$, and $\{J_1, \ldots, J_{i_0}\} \subset \partial H_1 - \Delta$, and $\{J_{i_0+1}, \ldots, J_{n-1}\} \subset \partial H_2$. By assumption (I)($n-i$), $C_1 = \{J_1, \ldots, J_{i_0}\}$ does not bind $F_{k_1} = \pi_1(H_1)$ of F_k, therefore from Lemma 2.6 we know that $\partial H_1 - C_1$ is compressible in H_1 (hence in H), and by Assertion 2, each compressing disk of $\partial H_1 - C_1$ (after a small isotopy, if necessary) is a compressing disk of $\partial H - (C - J_n)$ in H, and by Assertion 2, each compressing disk of $\partial H_1 - C_1$ separates H (therefore H_1). After applying this argument finitely often, we can reduce to the following situation (say): $J_1 \subset$ a handlebody H', cut out from H, with genus of $H' = k' = k - n + 2 > 0$. By assumption (I)($n-1$), J_1 does not bind the free factor $F_{k-n+2} = \pi_1(H')$ of $\pi_1(H)$, and again by Lemma 2.6 and Assertion 1, we obtain a compressing disk of $\partial H' - J_1$ which is also a compressing disk of $\partial H - (C - J_n)$. By Assertion 2, a compressing disk of $\partial H' - J_1$, say Δ', divides H'

into two handlebodies H_1' and H_2' with positive genus. Suppose $J_1 \subset \partial H_1'$, then $\partial H_2'$ has a nonseparating compressing disk in H_2', which is also a nonseparating compressing disk of H (after a small isotopy, if necessary). This contradicts Assertion 2, which followed from the assumption condition (*), so we have a contradiction to (*).

Thus we have finished the proof. □

Remark. From the proof of Theorem 3.1 we know that without assumption (II), we can always choose a nonseparating compressing disk Δ of $\partial H - (C - J_n)$ and obtain S^* as before with $\partial S^* = \{J_1', \dots, J_{i_0}', J_{i_0+1}', J_{i_0+1}'', \dots, J_{j_0}', J_{j_0}''\} \cup \partial\Delta$, say, where $1 \leq i_0 < j_0 \leq n-1$. It is not hard to show that J_1 bounds a disk on ∂H_{C-J_1}, therefore $H_C = H_{C-J_1} \# D^3$, where $\#$ denotes the connected sum, in other words, H_C can be obtained from H_{C-J_1} by removing an open 3-cell in the interior of H_{C-J_1}, hence H_C has incompressible boundary if and only if H_{C-J_1} does, unless $H_{C-J_1} = D^3$. But for H_{C-J_1}, assumption (I)(1)...$(n-1)$ cannot guarantee that H_{C-J_1} has incompressible boundary, since there exist examples (see [6]) which show that none of the conditions in Przytycki's conjecture can be deleted. Thus assumption (II) in Theorem 3.1 is needed.

Acknowledgement

The author is very grateful to Professor He Baihe and Professor Sun Yifeng for their helpful advices in preparing this paper.

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